Optics Letters

Statistical dispersion relation for spatially broadband fields

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The dispersion relation is fundamental to a physical phenomenon that develops in both space and time. This equation connects the spatial and temporal frequencies involved in the dynamic process through the material constants. Electromagnetic *plane waves* propagating in *homogeneous* media are bound by simple dispersion relation, which sets the magnitude of the spatial frequency, *k*, as being proportional to the temporal frequency, *ω*, with the proportionality constant dependent on the refractive index, *n*, and the speed of light in vacuum, *c*. Here we show that, for *spatially broadband fields*, an analog dispersion relation can be derived, except in this case the *k*-vector variance is connected with the temporal frequency through the statistics of the refractive index fluctuations in the medium. We discuss how this relationship can be used to retrieve information about refractive index distributions in biological tissues. This result is particularly significant in measurements of angular light scattering and quantitative phase imaging of biological structures. © 2016 Optical Society of America

Propagation of light in inhomogeneous media is particularly important in the context of biomedical optical imaging [1,2]. Light scattering by tissues determines the overall quality of the imaging: too little scattering results in a low contrast of the intensity image, while too much scattering distorts the image altogether. Most commonly, in practice, we use two extreme approximations: the first-order Born approximation for weakly scattering objects (see, e.g., Chapter 13 in [3] or Chapter 2 in [4]) and the diffusion equation for the strongly scattering regimes (see, e.g., Chapter 13 in [5]). The former regime, applicable for thin, transparent objects, e.g., single unlabeled cells and tissue slices, allows for *coherent* field calculation and inverse scattering solutions at high resolutions, limited by the wavelength of light. The latter deals with completely spatially

incoherent fields and applies to imaging deeper in the tissue, but with much coarser resolution, limited by the transport mean free path.

It is an accepted fact in the literature that the refractive index of tissues can report on the onset and development of various diseases [6,7]. However, despite many years of research, measuring the refractive index distribution of tissues over broad spatial scales, e.g., across an entire organ, in both normal and disease states, remains a grand challenge in biomedical optics [2].

Here, we derive a theoretical relationship between the statistics of the wave vector distribution and that of the refractive index of an inhomogeneous medium. This relationship can be used in experiments of light scattering or quantitative phase imaging (QPI) [4] to obtain information about the refractive index spatial fluctuations.

Let us consider first the Helmholtz equation:

$$\nabla^2 U(\mathbf{r}, \omega) + n^2 \beta_0^2 U(\mathbf{r}, \omega) = 0, \tag{1}$$

where U is the field in a medium, n is the refractive index of the medium, and β_0 is the wavenumber in vacuum, $\beta_0 = \omega/c$. Note that, if the medium is homogeneous, i.e., n is independent of $\mathbf{r}$, taking the Fourier transform with respect to $\mathbf{r}$ of in Eq. (1) leads to

$$(-k^2 + n^2 \beta_0^2) U(\mathbf{k}, \omega) = 0, \tag{2}$$

where $\mathbf{k}$ is the wave vector and $k = |\mathbf{k}|$. Equation (2) has non-trivial solutions only when the well-known dispersion relation is fulfilled, namely, $k^2 = n^2 \beta_0^2$.

If the medium is inhomogeneous [Fig. 1(a)], Fourier transforming Eq. (1) does not yield a simple dispersion relation, but, instead,

$$-k^2 U(\mathbf{k}, \omega) + \beta_0^2 n^2(\mathbf{k}) \circledcirc_\mathbf{k} U(\mathbf{k}, \omega) = 0, \tag{3}$$

where $\circledcirc_\mathbf{k}$ represents the convolution operator in the wave vector space $\mathbf{k}$. Nevertheless, we show that, in this case, we can obtain a statistical dispersion relation, i.e., an equation that relates spatially averaged fluctuations of the wave vector and the refractive index.

Let us rewrite Eq. (1) in such a way as to have the spatially varying part of the refractive index on the right-hand side, acting as a secondary source term,
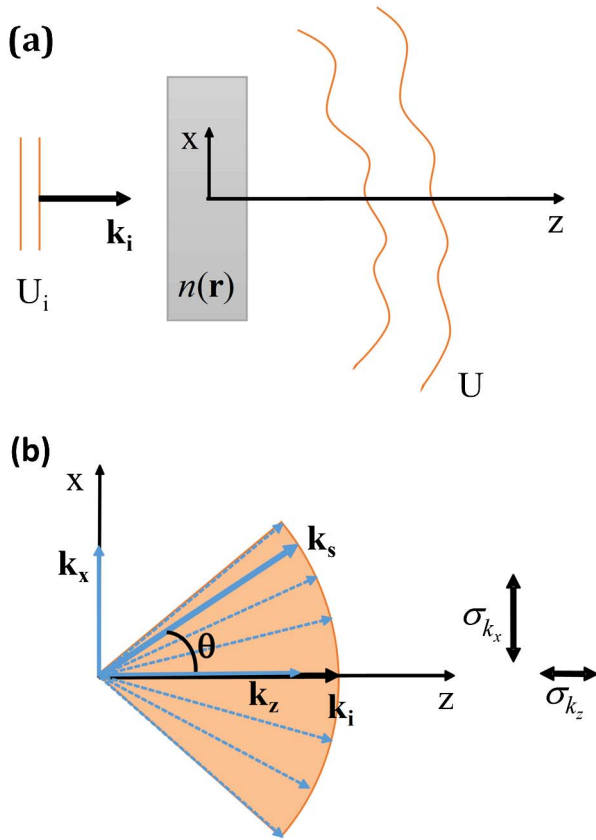
Fig. 1. (a) Scattering geometry. (b) Distribution of **k**-vectors in a scattering experiment, $\mathbf{k}_i$ and $\mathbf{k}_s$ are, respectively, the incident and scattered wave vector. σ_{k_x} and σ_{k_z} represent the spread (standard deviation) of the wavevector transverse and axial components, respectively.

$$\nabla^2 U(\mathbf{r}, \omega) + n_0^2 \beta_0^2 U(\mathbf{r}, \omega) = -[n^2(\mathbf{r}) - n_0^2]\beta_0^2 U(\mathbf{r}, \omega), \quad (4)$$

where $n_0^2 = \frac{1}{V}\int n^2(\mathbf{r}) d^3\mathbf{r} = \langle n^2(\mathbf{r}) \rangle$, and V is the volume of the medium. To obtain spatially averaged information about the field propagation in the inhomogeneous medium, we multiply Eq. (4) by U^* and average over the volume of interest,

$$\langle \nabla^2 U(\mathbf{r}, \omega) U^*(\mathbf{r}, \omega) \rangle_{\mathbf{r}} + n_0^2 \beta_0^2 \langle U(\mathbf{r}, \omega) U^*(\mathbf{r}, \omega) \rangle_{\mathbf{r}}$$
$$= -\beta_0^2 \langle [n^2(\mathbf{r}) - n_0^2] U(\mathbf{r}, \omega) U^*(\mathbf{r}, \omega) \rangle_{\mathbf{r}}. \quad (5)$$

In Eq. (5), $\langle \rangle_{\mathbf{r}}$ stands for spatial average, defined as

$$\langle f(\mathbf{r})g^*(\mathbf{r}) \rangle_{\mathbf{r}} = \frac{1}{V}\int f(\mathbf{r})g^*(\mathbf{r})d^3\mathbf{r}. \quad (6)$$

We note that the volume integral in the spatial average [right-hand side of Eq. (6)] can be expressed in terms of a spatial correlation evaluated at the origin, namely,

$$\int f(\mathbf{r}')g^*(\mathbf{r}' + \mathbf{r})d^3\mathbf{r}'\Big|_{\mathbf{r}=0} = [f \otimes g]|_{\mathbf{r}=0}, \quad (7)$$

where $\otimes$ stands for the spatial correlation operator. Using the central ordinate theorem, we can express the correlation at the origin as an integral over the **k**-space,

$$[f \otimes g]|_{\mathbf{r}=0} = \frac{1}{(2\pi)^3}\int \tilde{f}(\mathbf{k})\tilde{g}(\mathbf{k})d^3\mathbf{k}, \quad (8)$$

where "~" designates the Fourier transform. Combining Eqs. (7) and (8), we can express the spatial averages in Eq. (5) in terms of **k**-space integrals, as

$$-\frac{1}{(2\pi)^3}\int k^2 |U(\mathbf{k})|^2 d^3\mathbf{k} + \frac{n_0^2\beta_0^2}{8\pi^3}\int |U(\mathbf{k})|^2 d^3\mathbf{k}$$
$$= -\beta_0^2 \int [n^2(\mathbf{r}) - n_0^2]|U(\mathbf{r}, \omega)|^2 d^3\mathbf{r}. \quad (9)$$

Let us divide both sides of Eq. (9) by $\frac{1}{(2\pi)^3}\int |U(\mathbf{k}, \omega)|^2 d^3\mathbf{k}$ to obtain

$$-\langle k^2 \rangle + n_0^2\beta_0^2 = \frac{-\beta_0^2 \int [n^2(\mathbf{r}) - n_0^2]|U(\mathbf{r}, \omega)|^2 d^3\mathbf{r}}{\int |U(\mathbf{r}, \omega)|^2 d^3\mathbf{r}}. \quad (10)$$

In arriving at Eq. (10), we use Parseval's theorem, $\int |U(\mathbf{k}, \omega)|^2 d^3\mathbf{k} = (2\pi)^3 \int |U(\mathbf{r}, \omega)|^2 d^3\mathbf{r}$.

Note that $\langle k^2 \rangle$ is the ensemble-averaged k^2, defined using the normalized power spectrum, p, as probability density,

$$p(\mathbf{k}, \omega) = \frac{|U(\mathbf{k}, \omega)|^2}{\int |U(\mathbf{k}, \omega)|^2 d^3\mathbf{k}}. \quad (11)$$

The right-hand side of Eq. (10) can be simplified using the weak scattering approximation. Under the first-order Born approximation, the field on the right-hand side is replaced by the incident field, assumed to be a plane wave along z, $U_i(\mathbf{r}, \omega) = A(\omega) \exp[in_0\beta_0 z]$.

Thus, Eq. (10) becomes

$$-\langle k^2 \rangle + n_0^2\beta_0^2 = -\frac{1}{V}\beta_0^2 \int [n^2(\mathbf{r}) - n_0^2]d^3\mathbf{r} = -\beta_0^2\sigma_n^2. \quad (12)$$

In Eq. (12), $\sigma_n^2 = \langle n^2(\tilde{\mathbf{r}}) \rangle - n_0^2$ represents the *variance* of the spatial distribution of the refractive index. Finally, we obtain the *statistical dispersion relation* for a field in weakly scattering medium, namely,

$$\langle k^2 \rangle = n_0^2\beta_0^2\left(1 + \frac{\sigma_n^2}{n_0^2}\right). \quad (13)$$

Equation (13) represents the main result of this Letter. It establishes the relationship between the second-order moment of the **k**-vector, $\langle k^2 \rangle = \langle k_x^2 \rangle + \langle k_y^2 \rangle + \langle k_z^2 \rangle$, and the statistics of the refractive index fluctuations. Clearly, when $\sigma_n \to 0$, we recover the homogeneous dispersion relation, $\langle k^2 \rangle = n_0^2\beta_0^2$.

Experimentally, one can gain access to $\langle k^2 \rangle$ via angular scattering measurements on the weakly scattering sample [Fig. 1(b)]. In particular, for far-zone measurements in which the small angle approximation applies, we can assume $k_z \approx n_0\beta_0$ and $\langle k_z^2 \rangle = n_0^2\beta_0^2$. Thus, Eq. (13) yields

$$\langle k_\perp^2 \rangle = \beta_0^2\sigma_n^2. \quad (14)$$

Assuming symmetry around the z axis, i.e., $\langle \mathbf{k}_\perp \rangle = 0$, means that $\langle k_\perp^2 \rangle = \langle k_x^2 \rangle + \langle k_y^2 \rangle$ represents the variance of the transverse components of the **k**-vector. The transverse **k**-vector can be expressed in terms of the scattering angle, θ, via $k_\perp = n_0\beta_0 \sin(\theta)$. Thus, measuring the power spectrum of the scattered field gives direct access to the variance of the sample refractive index. Such measurements can be valuable in assessing the correlation between the refractive index of thin tissues (e.g., biopsies) and diseased states. While measuring traditional (goniometer-based) angular scattering from such transparent specimens is challenging due to extremely low scattered signals, it has been shown that quantitative phase imaging can be used effectively instead [8–11]. For example, using QPI, it

has been observed that higher variance of the refractive index correlates with tumors in prostate biopsies [6]. Recently, it has been also shown that scattering parameters derived from QPI measurements hold valuable information of pathological relevance [12,13].

There is another application of Eq. (14), which is in terms of scattering effects on the spatial coherence of the field, as follows. If the spatial bandwidth, defined as standard deviation, $\sigma_{k_\perp} = \sqrt{\langle k_\perp^2 \rangle}$, *increases*, the width of the transverse spatial correlation *decreases* accordingly, due to the Fourier relationship between the spatial correlation and power spectrum,

$$\frac{1}{A} \int U(\mathbf{r}'_\perp, \omega) U^*(\mathbf{r}_\perp - \mathbf{r}'_\perp, \omega) \mathrm{d}^3 \mathbf{r}'_\perp = |U(\mathbf{k}_\perp, \omega)|^2, \quad \textbf{(15)}$$

where A represents the area of interest. As shown in Ref. [14], we can define the transverse coherence length as $\sigma_{\mathbf{r}_\perp} = 2\pi / \sigma_{\mathbf{k}_\perp}$, such that $\sigma_{\mathbf{r}_\perp}^2$ represents the *coherence area* of the field. It follows that Eq. (14) connects the coherence area to the second-order statistics of the refractive index, namely,

$$\sigma_{\mathbf{r}_\perp}^2 = \frac{\lambda^2}{\sigma_n^2}. \quad \textbf{(16)}$$

Equation (16) has a simple interpretation: it captures the loss of spatial coherence as the field propagates though a scattering medium of refractive index variance, σ_n. In the absence of scattering, $\sigma_n = 0$, the coherence area becomes that of the incident plane wave, i.e., infinitely large. This result was obtained under the small angle approximation, so it does not cover the strongly scattering regime. However, in the case of diffusion, we anticipate that the coherence area reaches its minimum value, $\sigma_{\mathbf{r}_\perp}^2 \cong \lambda^2$.

In sum, we derived a statistical dispersion relation for waves propagating in inhomogeneous media. The relationship between the statistics of the wave vector, which is measurable, and the refractive index provides an opportunity for angular scattering and QPI measurements. The main result of our calculation, Eq. (13), was obtained within the Born approximation. This formalism applies for weakly, singly scattering media, such as single, unlabeled cells and thin slices of untagged tissues. These particular systems can offer a broad range of applications, from basic cell studies to clinical application. One exciting application is to correlate the refractive index variance with diseases such as cancer, which offers opportunities for label-free diagnosis.

The calculations here assumed a static medium. However, an interesting extension of these results will be to treat the dynamic light scattering problem, in which the refractive index distribution is not only inhomogeneous, but also time dependent.

REFERENCES

1. W. C. Chew, *Waves and Fields in Inhomogeneous Media* (IEEE, 1995).
2. V. V. Tuchin and Society of Photo-optical Instrumentation Engineers, *Tissue Optics: Light Scattering Methods and Instruments for Medical Diagnosis* (SPIE, 2007).
3. M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light* (Cambridge University, 1999).
4. G. Popescu, *Quantitative Phase Imaging of Cells and Tissues* (McGraw-Hill, 2011).
5. J. G. Fujimoto and D. L. Farkas, *Biomedical Optical Imaging* (Oxford University, 2009).
6. Z. Wang, K. Tangella, A. Balla, and G. Popescu, J. Biomed. Opt. **16**, 116017 (2011).
7. M. Hunter, V. Backman, G. Popescu, M. Kalashnikov, C. W. Boone, A. Wax, G. Venkatesh, K. Badizadegan, G. D. Stoner, and M. S. Feld, Phys. Rev. Lett. **97**, 138102 (2006).
8. T. Kim, R. Zhou, M. Mir, S. D. Babacan, P. S. Carney, L. L. Goddard, and G. Popescu, Nat. Photonics **8**, 256 (2014).
9. Z. Wang, H. Ding, and G. Popescu, Opt. Lett. **36**, 1215 (2011).
10. H. Ding, Z. Wang, X. Liang, S. A. Boppart, K. Tangella, and G. Popescu, Opt. Lett. **36**, 2281 (2011).
11. Y. Jo, J. Jung, M.-H. Kim, H. Park, S.-J. Kang, and Y. Park, Opt. Express **23**, 15792 (2015).
12. S. Sridharan, V. Macias, K. Tangella, A. Kajdacsy-Balla, and G. Popescu, Sci. Rep. **5**, 9976 (2015).
13. H. Majeed, M. E. Kandel, K. Han, Z. Luo, V. Macias, K. Tangella, A. Balla, and G. Popescu, J. Biomed. Opt. **20**, 111210 (2015).
14. T. Kim, R. Zhu, T. H. Nguyen, R. Zhou, C. Edwards, L. L. Goddard, and G. Popescu, Opt. Express **21**, 20806 (2013).